The completion of a metric space

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N.B. This has not yet been carefully proofread. Also, the argument can be simplified.

Entrance. Fix a metric space (X, m); our goal is to construct its (metric) *completion* $(\widehat{X}, \widehat{\mathsf{m}})$.

Call a sequence $\mathbf{x} \subset X$ a ***blip*** if $\mathbf{x}$ is m-Cauchy, and let Ω denote the set of blips. *Exercise: Given blips* $\mathbf{x}$ *and* $\mathbf{y}$, *the seq* $[n \mapsto \mathsf{m}(x_n, y_n)]$ *is* $\mathbb{R}$*-cauchy.* So

$$\mu(\mathbf{x}, \mathbf{y}) \; := \; \lim_{n\to\infty} \mathsf{m}(x_n, y_n)\,;$$

is well-defined, hence is a pseudo-metric on Ω.

The rest of this note will show that (Ω, μ) is a *complete* pseudo-metric space. (Once shown, the remaining argument is straightforward. Say two blips are ***equivalent*** if $\mu(\mathbf{x}, \mathbf{y}) = 0$, and let $\widehat{X}$ be the set of equivalence-classes. Let $\widehat{\mathsf{m}}$ be the metric on $\widehat{X}$ arises from applying μ to representatives of the equivalence-classes. Automatically $\widehat{\mathsf{m}}$ is a complete metric, since μ is complete (as a pseudo-metric).)

I will use $\mathbf{b}, \mathbf{x}, \mathbf{y}$ for blips; elements of Ω.

A subsequence $\mathbf{x}'$ of a blip $\mathbf{x}$ will be called a ***subblip***; and (1a) justifies this term.

1: Proposition. *Suppose* $\mathbf{x}$ *is* m*-cauchy. Then:*

a: *Each subsequence* $\mathbf{x}'$ *of* $\mathbf{x}$ *is* m*-cauchy, and* $\mu(\mathbf{x}', \mathbf{x}) = 0$.

b: *Given numbers* $\varepsilon_K \searrow 0$ *there exists a subblip* $(z_j)_{j=1}^{\infty}$ *of* $\mathbf{x}$ *st.* $\forall K$: $\mathsf{m}\text{-Diam}\big((z_j)_{j=K}^{\infty}\big) < \varepsilon_K$. ◊

Proof. Left to reader. ♦

2: Basic Lemma. *Suppose* $(\mathbf{y}^n)_{n=1}^{\infty} \subset \Omega$ *is a sequence* (not.nec μ-Cauchy) *of blips. Write each* $\mathbf{y}^n$ *as* $(y_j^n)_{j=1}^{\infty}$. *Then for each* n *we can drop to a subsequence of* $\mathbf{y}^n$ *so that now the following holds for all* N.

(Property[N]): *For all* $K \in [1\,..\,N)$ *and all* $j \geq N$:

$$\mathsf{m}(y_j^K, y_j^N) \;\leq\; \frac{1}{4^N} + \mu(\mathbf{y}^K, \mathbf{y}^N)\,. \qquad (3) \quad ◊$$

Proof. We do this inductively on N. At stage N, for $K = 1, \ldots, N{-}1$, we will drop to a subsequence of $\mathbf{y}^K$ (and renumber its terms), *but* will not change the first $N{-}1$ terms of $\mathbf{y}^K$. Thus this iterative subsequencing operation does indeed leave us with subsequences after N has gone to ∞.

At stage N: For $K = 1, 2, \ldots, N{-}1$, let J_K be large enough that (3) holds for all $j \geq J_K$. Let

$$J \;:=\; \operatorname{Max}(N, J_1, \ldots, J_{N-1})\,.$$

Now for K going from 1 upto and *including* N, discard *all* terms

$$y_N^K, y_{N+1}^K, y_{N+2}^K, \ldots, y_{J-1}^K$$

from the sequence $\mathbf{y}^K$ (and renumber). Now Property[N] holds. Further dropping to subsequences at later stages will preserve Property[N]. ♦

Proof that (Ω, μ) is complete

Fix a μ-Cauchy blip-seq $(\mathbf{y}^n)_{n=1}^{\infty}$. We will construct a blip $\mathbf{b}$ st. $[\mu\text{-}\lim_{n\to\infty} \mathbf{y}^n] = \mathbf{b}$.

Proof. Courtesy (1a), ISTProve this convergence for some subsequence of $(\mathbf{y}^n)_{n=1}^{\infty}$. By (1b) we can drop to a subsequence (and renumber) so that now

$$\text{For all } K \leq n\text{:} \qquad \mu(\mathbf{y}^K, \mathbf{y}^n) \;<\; \frac{1}{2^K}\,. \qquad (4)$$

Moreover, by (1a) we can replace each $\mathbf{y}^n$ by a subblip —an (1b) says there are subblips so that

$$\operatorname{Diam}(\mathbf{y}^n) \;<\; \frac{1}{3^n}\,. \qquad (5)$$

As our last preparation, enumerate $\mathbf{y}^n$ as $(y_j^n)_{j=1}^{\infty}$.

A μ-limit of $(\mathbf{y}^n)_{n=1}^{\infty}$. Define a seq $\mathbf{b} = (b_n)_{n=1}^{\infty}$ by $b_n := y_n^n$. To see that $\mathbf{b}$ is m-Cauchy, note that

$$\begin{aligned}
\mathsf{m}(b_{n-1}, b_n) \;&\leq\; \mathsf{m}(b_{n-1}, y_n^{n-1}) + \mathsf{m}(y_n^{n-1}, b_n) \\
&\leq\; \operatorname{Diam}(\mathbf{y}^{n-1}) + \Big[\frac{1}{4^n} + \mu(\mathbf{y}^{n-1}, \mathbf{y}^n)\Big]\,, \quad \text{by Property}[n], \\
&\leq\; \frac{1}{3^{n-1}} + \Big[\frac{1}{4^n} + \frac{1}{2^{n-1}}\Big]\,, \qquad \text{by (5) and (4).}
\end{aligned}$$

This last sum is a summable function of n. Thus $\mathbf{b}$ is m-Cauchy.

Establishing $\mathbf{y}^K \xrightarrow{\mu} \mathbf{b}$ as $K \nearrow \infty$. Fix K. For each $n > K$ note that

$$\begin{aligned} \mathsf{m}(y_n^K, b_n) &\le \frac{1}{4^n} + \mu(\mathbf{y}^K, \mathbf{y}^n)\,, \quad \text{by Property}[n], \\ &\le \frac{1}{4^n} + \frac{1}{2^K}\,, \qquad \text{by (4)}, \\ &\le \frac{1}{2^{K-1}}\,, \qquad \text{since } n > K. \end{aligned}$$

Sending $n \nearrow \infty$, then, gives $\mu(\mathbf{y}^K, \mathbf{b}) \le \frac{1}{2^{K-1}}$. And this last number goes to zero as $K \nearrow \infty$. ♦

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